# Coulomb branches and G-symmetry of low dim. TQFT

$\mathcal{N}=4$ 3D gauge theory

Prehistory:

A. Twistings and Brauer group of $H^*, K^*$ of BG
B. GLSM in 2 dims

1 Toda spaces & G-categories
2. Coulomb branch for polarized reps $V \oplus V^*$
3. Monopoles w/ singularities
4. Quantum & Symplectic Cohomology
5. Misc.

Old story: $(G, h)$ 3D CS theory
reduced on $S^1 \to$ 2D TQFT (Verlinde)
space = Nonabelian Theta functions
$\Gamma(\mathrm{Bun}_G(\Sigma); \mathcal{O}(h))$

Topological calculation of the TQFT

$G$, $h \in H^4(BG; \mathbb{Z}) = $ factor in $\mathbb{B}^2 GL_1 K^0(BG)$ $\quad K(\mathbb{Z};2)$ "

$GL_1 K = BU(1) \times (\text{higher grps})$

gen coh $\mathcal{H}$: $\alpha \in H^1(X; GL(\mathcal{H})) \to$ twisted coeff system $/X$

can compute $H^*(X; \mathcal{H}(\alpha))$

$\beta \in H^2(X; GL(\mathcal{H}))$ curving, exp. curvature.

$$\Sigma \times \mathrm{Bun}_G(\Sigma) \xrightarrow{E} BG$$

$$E^*(h) \in H^4(\Sigma \times \mathrm{Bun}_G(\Sigma)) \xrightarrow{\int_\Sigma} H^2(\mathrm{Bun}_G(\Sigma); \mathbb{Z}(1)).$$

TQFT $\quad S^1 \to$ space $\qquad S^1 \times \mathrm{Bun}_G(S^1) \xrightarrow{e} BG$

$K_G(G; h)) \qquad \int_{S^1} e^* h \in BGL_1(K)(G/_G)$

Other path to same TQFT

$$\frac{\qquad N(h) \qquad}{\text{3D } \mathcal{N}=4 \ G\text{-theory}}$$
$$\overline{\qquad \mathcal{N} \qquad}$$

twisted Neumann condition

$\in K^4(BG)$

Fact: $h$ defines a twisted Neumann cond

$H^2(BG; \mathbb{C}^\times)$

$GL(K)$)

<u>2D GLSM</u> Witten: integration formulas for classes in $H^\bullet(\mathrm{Bun}_G(\Sigma))$

"B-model construction of gauge theory".

$\omega \in H^2(\mathrm{Bun}_G(\Sigma); \mathbb{Z})$

Formula: $h \cdot \omega + \Phi$ (tautological generators)

$$\Sigma \times \mathrm{Bun}_G(\Sigma) \xrightarrow{E} BG$$

$\int_\Sigma E^* c p_k$ $p_k$ generators of $H^*(BG)$

(Atiyah-Bott gens) for $H^\bullet(\mathrm{Bun}_G \Sigma)$

$$\int_{\mathrm{Bun}_G(\Sigma)} \exp(h\omega + \Phi(\text{tautological}))$$

$\xi \in \mathfrak{g}_W$

Formula: $\sum_{\text{reg elts of } G} h^{\text{power}} \cdot$ Hessian det of $\Big(\overbrace{\tfrac{1}{2} h \|\xi\|^2 + \Phi(\xi)}^{}\Big)^{1-g}$ at crit point of

$\overline{\Phi}$ : invar function on $\mathfrak{g}$

quadratic + nilpotent

Critical point of $\overline{\Phi}$ : $d\overline{\Phi} \in$ coroot lattice

$\mathfrak{g} \to \mathfrak{g}^*$

$\mathfrak{g}/\!/G = \mathfrak{t}/W$

TQFT for B-model of $\Phi$ on $\mathfrak{t}/W$.

Special case: choose $G$-rep. $V$

$$\begin{array}{c} E^*V \\ \downarrow \\ \Sigma \times \mathrm{Bun}_G(\Sigma) \end{array} \xrightarrow{\pi} \mathrm{Bun}_G(\Sigma)$$

$c.\ \mathrm{Index}(\Sigma; E^*V \otimes K^{1/2})$

$= \sum \mu^{-k} c_k(\mathrm{Index})$ — formally near $\mu = \infty$

↳ mass parameter

<u>Remark</u> $c_{total}$ = euler (bundle + scaling $\mathbb{C}^*$ action)

Here $\Phi(\Sigma) = \frac{k}{2}\|\Sigma\|^2 + \mathrm{Tr}_V\big((\Sigma+\mu)(\log(\Sigma+\mu)-1)\big)$

superpotential for GLSM)

B/ $H^*(BG)$)

(Woodward) K-theoretic formula

$H^*(BG) = \mathbb{C}[\mathfrak{t}/W]$ $\qquad \Sigma \longrightarrow x = e^{\Sigma}$

$K_G(pt) = \mathbb{C}[T/W]$

$m = e^{\mu}$

superpotential: $\boxed{\frac{k}{2}(\log x)^2 + \mathrm{Tr}_V\, \mathrm{Li}_2(m^{-1}x)}$

B side

2D mirror symmetry

Interpretation of $\int \mathrm{euler}_{S^1}(\text{index bundle})$

$\int_{\text{vect space} = \mathrm{Ind}(V)} (\text{coh class})$ nonsense

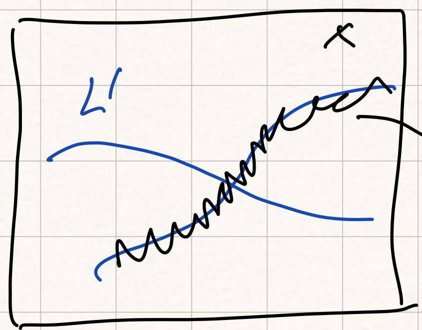
W. G action: $\int_{\text{Vect space}} (\text{coh class}) = \dfrac{\text{coh class}}{e_{G\times S^1}(\mathrm{Ind}(V))}$

Question: Closed GW invariants of $V$ gauged by $G$

$\mathrm{Maps}(\Sigma; V/G)$ stack.

Kapustin Rozansky Saulina 2-cat for a holomorphic sympl manifold

conjectural generator for 3D RW theory




$L \subset X$ hol Lagrangian

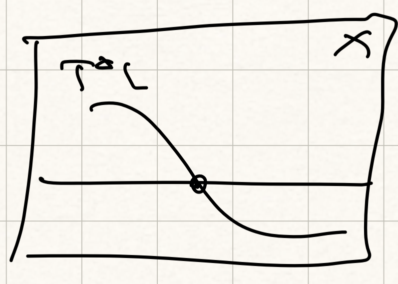
$\cong T^*L$

$L' = \Gamma(d\Phi)$

Hom $:= MF(L; \Phi)$.

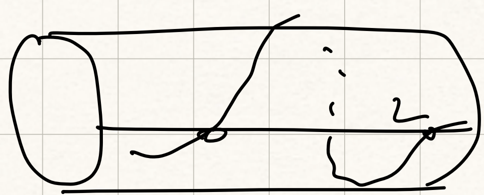
(Orlov et al)

Special case:




$\varphi : L \to \mathbb{C}$

$L$ $\quad L' : \Gamma(d\varphi)$




Hom in periodized $T^*L$: computed when $d\varphi \in$ integer lattice.

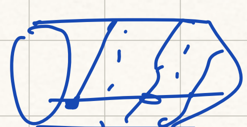
Witten + K Hung improvement:

$T^* T_{\mathbb{C}} \longrightarrow T_{\mathbb{C}}^\vee \times T_{\mathbb{C}}$ take $\Gamma(\exp d\varphi)$ inside

Hom$(\Gamma;$ unit $1 \times T)$

$=$ 2d TQFT giving Witten, Verlinde & generalized formulas

Eg $\quad h = \frac{1}{2}|z|^2 \qquad h : T_{\mathbb{C}} \to T_{\mathbb{C}}^\vee$



Nonab case: $(T_{\mathbb{C}}^\vee \times T_{\mathbb{C}})/W$, "volume form": volume of conj class

$\longrightarrow$ get K theory Witten formula

Our catch: Must exclude singular points in $T_{\mathbb{C}}$.

Space Defined by Bezrukavnikov, Mirkovic, Finkelberg

$H^*$ Spec $H_*^G(\Omega G)$

$K^*$ Spec $K_*^G(\Omega G)$

Resol of singularities of $(T \times T^\vee)/W$

- affine
- group structure over $T/W$ (Hopf structure)
- Holomorphic symplectic

$\uparrow$

Toda Integrable system [Givental] $J(G)$

— $L(\mathfrak{g}, V)$ —

RW (Toda space)

— unit —

Rmk 2 category has a sym mon structure bc of fibrewise abelian structure

3D $N=4$ TQFT $\Longleftrightarrow$ RW theory of Toda space

Bdry conds

$=$ 2D TQFTs with G action $\Longleftrightarrow$ Ob in RWS (Toda space)

eg: smooth Lags with a local system of categories on them

Thm X compact Sympl G action

$QH_G^*(X)$ sheaf over $J(G)$ with Lag. support

$\mathcal{F}(X)$ extends to a sheaf of cat over $\uparrow$

(Conj: $HH^* \mathcal{F}(X) = QH^*(X)$)
$X$ compact

Braverman Nakajima Finkelberg

Birational modification of $J(G)$ associated to $V$ $(V \oplus V^*)$

<u>Obs</u> $V$ not compact. $SH^*_c(V)$ (Conj $= HH^*(2\mathcal{F}(V))$)

$QH^*_c(V)$

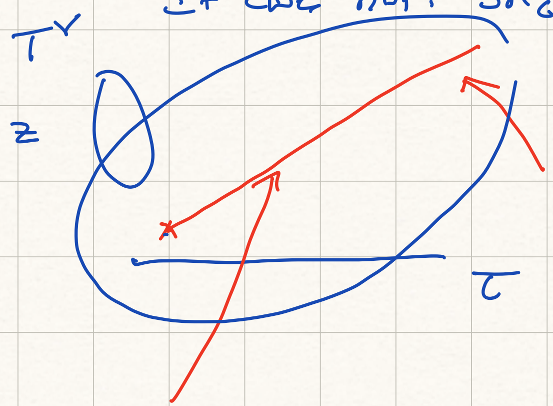
In many interesting cases: $SH^*_c(V)$ is a localization of $QH^*_c(V)$

<u>Eg</u> $V = \mathbb{C}$, $G = \mathbb{C}^\times$ $QH^*_c(V) = H^*(BS^1) = \mathbb{C}[\tau]$

$SH^*_c(V) \equiv \mathbb{C}[\tau, \tau^{-1}]$

How does $QH^*_c(V)$ sit in the Toda space?

It does not. $SH^*_c(V)$ support $= z = \tau$



$\exp(d\Psi)$

superpotential of GLSM

$\mathfrak{t} \longrightarrow T^\vee \xrightarrow{p} \mathbb{C}^\times$

$p \in \mathrm{Hom}(T^\vee; \mathbb{C}^\times)$

$= \pi_1 T$

- Euler class of Atiyah index bdle
- $\exp$ (d $\underbrace{\text{GLSM superpotential}}_{\text{Brave class.}}$) $p \circ \exp(d\Psi) =$ euler class of the index bdle of $V \longrightarrow \mathbb{P}^1$

$p$

<u>Thm</u> 1. $\mathrm{Coul}(G;V) =$ gluing 2 copies of $J(G)$ by vertical rational shift using $\exp(d\Psi)$

2. (Gonzalez, Mak, Pomerleano)

$\mathbb{C}[\mathrm{Coul}(G;V)] \subset \mathbb{C}\,J(G)$ preserves $QH^*_c(V) \subset SH^*_c(V)$.

3. $QK^*_C(X \times Y)$ lives in Lagrangian in Coulomb $(G;V)$

$SK^*_G(X;V)$ lives in $J(G)$.

Cor explicit computation from $T$, $W$, wts of $V$.

---

Monopoles w/ singularities: (Esfahani: $SU(2)$)

on 3-manifold

Monopoles: $G$ bdle with connection $A$ + $\Phi$, $\mathfrak{g}$-valued function

$$*F_A = d_A \Phi.$$

Fact on a compact mfld, $\Rightarrow$ $*F_A = d_A \Phi = 0$

($\Phi$ harmonic)

Want? nonzero examples

need singularities.

$$\|\Phi\| \sim \frac{k}{r} + m + O(r) \quad (r \text{ distance from sing pt})$$

Kronheimer: Get them from $S^1$-equivariant instantons on $\mathbb{R}^4$

$S^1$ acts $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$

$$\mathbb{R}^4/S^1 \overset{\text{homeo}}{\cong} \mathbb{R}^3$$

Instantons descend to singular monopoles.

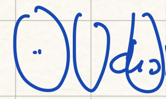
Coulomb: $H_*$ (space of monopoles w singularities)

sing type defined from $V$

solid pair of pants mult.

$\bar\partial$ $\quad d/dt$

$$\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$$

Idea (BFN) $\mathbb{P}^1$  $S^2$ $\longrightarrow$ formal $\mathbb{P}^1$  disk w doubled origin

$E_3$ structure

(moral version)

$V \longrightarrow$ Atiyah index bundle over $(\Omega G)_{\mathbb{C}}$

$\longrightarrow$ Coherent sheaf $(\otimes K^{1/2})$

$\longrightarrow$ Linear space (Gravitant)

$H_*(\mathrm{Thom}(P))$

Actual construction: De suspension of $\mathrm{Bun}_G(\,\vdots\,)$

by a space w/ increasing dim

mult on $H_*$, $K_*$ can be defined.

$H \neq V \oplus V^*$