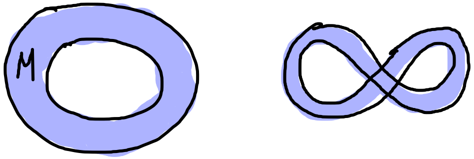
Topological quantum states

(Daniel Ranard, Isaac Kim, Alexei Kitaev)

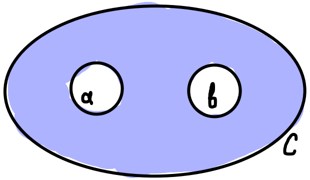
— We study "physical" objects such as spin Hamiltonians or quantum states on regions in $\mathbb{R}^d$

immersed manifolds with boundary



-- Quantum states are famously nonlocal. We identify a class of states that admit a local description (a bit like a sheaf but not quite).

-- The states that are consistent with the local data on a given region M ($\approx$ sheaf sections) form a convex set $\Sigma(M) = \mathrm{States}\,(\underbrace{\mathcal{A}(M)}_{\text{finite-dimensional } C^* \text{ algebra of "topological" operators}})$



$$\mathcal{A} = \bigoplus_{a,b,c} \mathbb{L}\left(V_c^{ab}\right), \qquad V_c^{ab} = \mathrm{Hom}\,(a,\, b \otimes c)$$

simple objects in a fusion category

-- We prove an "isomorphism theorem": $\Sigma(M)$ is invariant under deformations of M (or its boundary) $\Rightarrow$ isomorphism under regular homotopies

Quantum states basics

-- We consider states on f.d. C* algebras, which have the form $\mathcal{A}=\bigoplus_j \mathbb{L}(\mathcal{H}_j)$.
Usually, $\mathcal{A}=\mathbb{L}(\mathcal{H})$, so states are represented by density matrices.

-- A state ρ on $\mathbb{L}(\mathcal{H}\otimes\mathcal{L})$ can be <u>restricted</u> to $\mathbb{L}(\mathcal{H})$ (notation: $\rho|_{\mathbb{L}(\mathcal{H})}$ or $\mathrm{Tr}_{\mathcal{L}}\,\rho$)

by duality with the inclusion $\mathbb{L}(\mathcal{H})\to\mathbb{L}(\mathcal{H}\otimes\mathcal{L})$: $\left(\forall\ X\in\mathbb{L}(\mathcal{H})\quad \rho|_{\mathbb{L}(\mathcal{H})}(X)=\rho(X\otimes 1_{\mathcal{L}})\right)$

-- Every state σ on $\mathbb{L}(\mathcal{H})$ has a <u>purification</u>: a pure state $\overline{\sigma}$ on $\mathbb{L}(\mathcal{H}\otimes\mathcal{F})$ such that

$\overline{\sigma}|_{\mathbb{L}(\mathcal{H})}=\sigma$. All purifications are unitarily equivalent: $\overline{\sigma}''=\underline{(1_{\mathcal{H}}\otimes U)}\,\overline{\sigma}'\,\underline{(1_{\mathcal{H}}\otimes U)}^{\dagger}$.

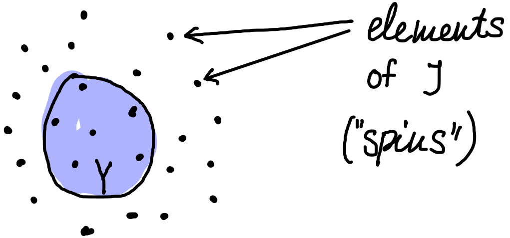
-- Let ρ be a state on $\mathbb{L}(\mathcal{H}\otimes\mathcal{L})$ s.t. $\rho|_{\mathbb{L}(\mathcal{H})}=\sigma$. Then $\overline{\rho}$ is unitarily equivalent to $\overline{\sigma}$.

Thus, $\mathcal{L}$ is identified with a
subsystem (tensor factor) of $\mathcal{F}$

$$\overline{\rho}\text{ on }\mathcal{H}\otimes\underbrace{\mathcal{L}\otimes\mathcal{M}}_{}\quad\xrightarrow{U}\quad\overline{\sigma}\text{ on }\mathcal{H}\otimes\underbrace{\mathcal{F}}_{}$$

General setting

-- Spin system $(\underset{\substack{\uparrow \\ \text{f.d. Hilbert space}}}{\mathcal{H}_x} : x \in \mathcal{J})$, $\mathcal{J} \subseteq X$ (usually $X = \mathbb{R}^d$)



-- For each region $Y \subseteq X$, $\mathcal{H}_Y = \bigotimes_{x \in \mathcal{J} \cap Y} \mathcal{H}_x$, $\underbrace{Alg_Y = \mathbb{L}(\mathcal{H}_Y)}_{\text{\underline{all} operators acting in region Y}}$,

-- For a (properly) immersed region M, spins are replicated.

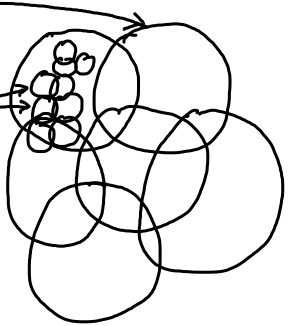
-- Need to choose 3 length parameters $r_0 \ll r_1 \ll r_2$, i.e. $r_0 \le c_1 r_1$, $r_1 \le c_2 r_2$

small constants

r_2 is the size of data patches (forming a coarse cover of X)

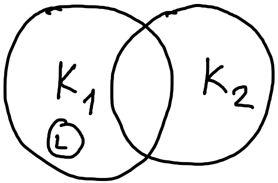
r_1 is the size of reference patches (finer cover)

r_0 enters some condition on local data



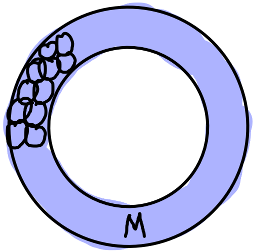
-- Local data: state σ_K on each data patch K (i.e. on the algebra Alg_K)

$$\sigma_{K_1}\big|_{K_1 \cap K_2} = \sigma_{K_2}\big|_{K_1 \cap K_2}$$



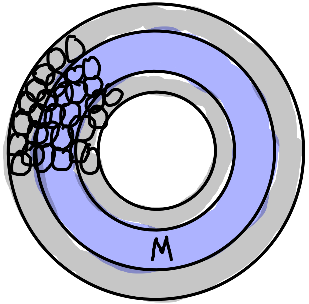
-- For each reference patch $L \subseteq K$, $\sigma_L = \sigma_K\big|_L$

Reference patches are used to define consistent states on bigger regions.



ρ is consistent with local data if

$\rho\big|_L = \sigma_L$ for all reference patches $L \subseteq M$



The set of strongly consistent states:

$2r_1$ - neighborhood of M

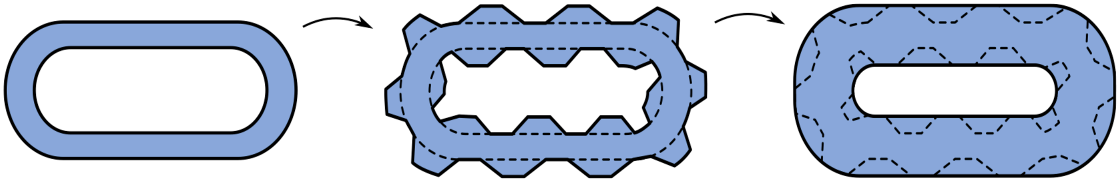
$$\Sigma(M) = \{ \rho \mid \exists\, \rho_+ \text{ on } M_+ \text{ consistent with } \rho \text{ and the local data} \}$$

-- To prove nice properties, we need consistency with the bigger data patches.

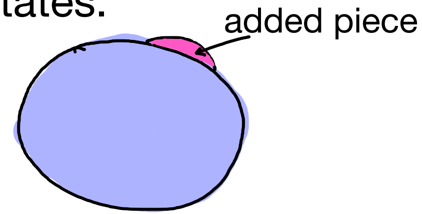
We also impose certain conditions on each

Extendibility condition

-- Purpose: allow for extension of compatible states to bigger regions

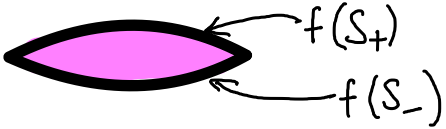


Each extension step (gate) and their composition are described by quantum channels, i.e. completely positive trace-preserving maps of states.



-- Geometry of extensions: we want something like this:

The added piece is the interior of an embedded sphere



$S_+ =$ northern hemisphere

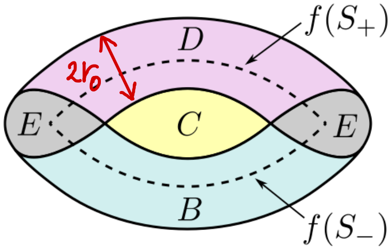
$S_- =$ southern hemisphere

$S_+ \cup S_- = S^{d-1}$

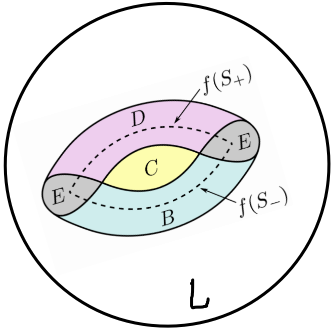
But:

a) We impose some geometric constraints in terms of Lipschitz constants

b) The sphere should be thickened by r_0



Extendibility in more detail



-- Conditions are imposed for a sufficiently large set of configurations within each reference patch L

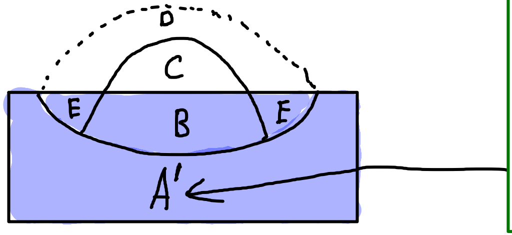
-- Each instance of the extendibility condition involves a purification $\bar{\sigma}$ of $\sigma_L|_{BCDE}$ using an abstract system A: $\bar{\sigma}$ is a pure state on $\mathbb{L}(\mathcal{H}_A \otimes \mathcal{H}_{BCDE})$ such that $\bar{\sigma}|_{BCDE} = \sigma_L|_{BCDE}$

-- The actual condition:

$\exists$ quantum channel $R_{BE \to BC}$

such that $\underline{(1_A \otimes R_{BE\to BC})[\bar{\sigma}|_{ABE}] = \bar{\sigma}|_{ABC}}$

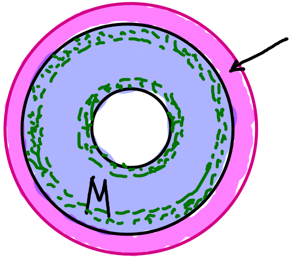
In applications of this condition, an existing region A' is identified with a part of A



Isomorphism theorem: general idea

-- If $M_1 \subseteq M_2$, then states on M_2 can be restricted to M_1 $T_{M_1,M_2} : \mathrm{States}_{M_2} \to \mathrm{States}_{M_1}$

-- Conversely, let M be "tame", i.e. having a suitably bounded boundary curvature and a sufficiently thick inner collar. Then local data-compatible states on M can be extended to $\tilde{M}$, which includes an outer collar.



outer collar attached to one boundary component

$$R\left[\underbrace{\Sigma(M)}\right] \subseteq \underbrace{\Sigma(\tilde{M})}$$

recovery channel

convex sets of strongly compatible states

-- The restrictions maps T_{M_1,M_2} form a contravariant functor from the category of immersed d-manifolds to quantum states

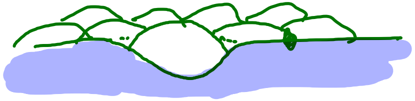
$$\begin{tikzcd} M_1 \arrow[rr, hook] \arrow[dr] & & M_2 \arrow[ll, bend right, dashed] \arrow[dl] \\ & \mathbb{R}^d & \end{tikzcd}$$

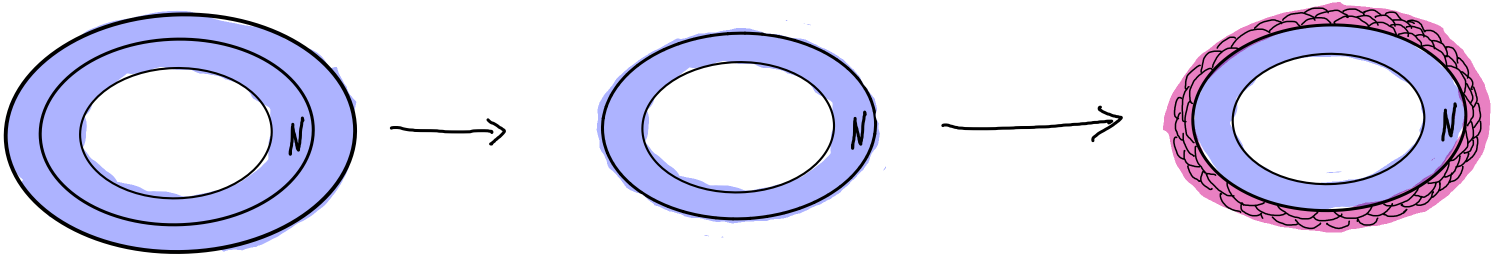
The recovery maps extend this functor to some localization of that category.

Uniqueness of extended state

-- Let N be a "tame" d-1 manifold immersed in $\mathbb{R}^d$, equipped with a normal vector and a tubular neighborhood, on which a consistent state ρ is defined.

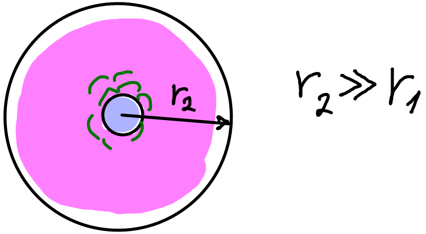
We erase the outer collar and partially regrow it (save $2r_1$).





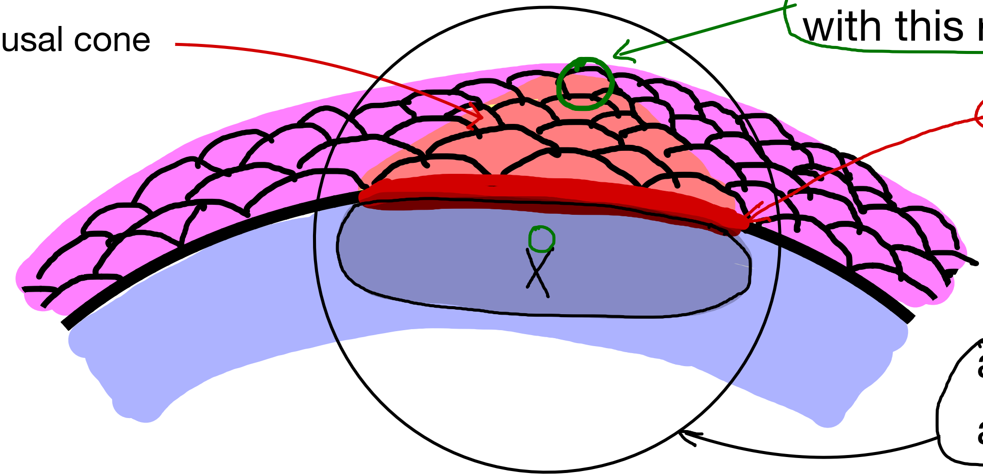
At each step, the current state is consistent with the reference patch containing the regions BCDE that are involved in the definition of the recovery channel. This guarantees that the recovered state is the same as before the erasure.

-- This way, we can grow a (slightly reduced) data patch from a small seed, or even from nothing



Existence of extension

-- When extending an arbitrary consistent state, the continued consistency requires a separate argument



causal cone

we want consistency with this reference patch

causal base

a data patch containing all relevant stuff

-- Let ρ be the initial state, defined on the blue region. Then $\rho|_X$ can be grown from nothing due to the uniqueness property.

-- We proceed by regrowing the causal base, and then grow the causal cone, including the reference patch in question.