Soft Intro to Kubota (2503.12618)

Collection of qtm systems of fixed "type" forms a symmetric monoidal category under "stacking"

Type: • Context — discrete : quantum lattice systems
— field Theory, wick-rotated

• Dimension d = spatial dimension

• Symmetry or Background fields (T)

$\Rightarrow$ notion of invertible qtm system

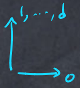
$\mathcal{I}_d(T)$ = moduli space of invertible systems

Kitaev Conjecture: $\{\mathcal{I}_d(T)\}_d$ form a spectrum

Intuition: $\mathcal{I}_d(T) \xrightarrow{f} \Omega\, \mathcal{I}_{d+1}(T)$

"Kitaev pump"

$\rho(Q)$, $Q \in \mathcal{I}_d(T)$ $\qquad \mathbb{1} \xrightarrow{\delta} Q \otimes Q^{-1}$



$\mathbb{1}$ $\qquad t = 0$

$\Big\downarrow \delta$

$\cdots (Q \otimes Q^{-1})(Q \otimes Q^{-1})(Q \otimes Q^{-1}) \cdots$ $\qquad t = 1/2$

$\Big\downarrow \delta^{-1}$ transposed

$\mathbb{1}$ $\qquad t = 1$

Context: Wick-rotated field Theory

T = sheaf $\mathcal{F}$ of background fields, $n = d+1$

$$F : \mathrm{Bord}_{\langle n-1, n\rangle}(\mathcal{F}) \longrightarrow t\mathrm{Vect}$$

Invertible topological Theories $\longleftrightarrow$ stable htpy Thy

$$\alpha : |\mathrm{Bord}_n(\mathcal{F})| \longrightarrow \mathcal{Q}$$

map of spectra

Get a "moduli space" = spectrum of such maps

Rmk: 1) Get a well-defined homotopy type

2) Nontopological invertible theories: differential refinement

3) Identify spectrum: (Anderson) dual to bordism spectrum

Kubota results: Context is qtm spin systems (∞ systems on $\mathbb{R}^d$)

T = bosonic, fermionic, w/internal cpt Lie gp G

1. Constructs moduli space $\mathcal{I}_d(T)$
2. Constructs Kitaev pump ρ & proves it's a hty $\simeq$
3. Computes low hty groups of $\mathcal{I}_\bullet(T)$, recovering known results
4. Describe associated homology theory ~ crystallographic symmetries

Moduli space of some "objects" Q:

$$\mathcal{F}: \text{"spaces"} \longrightarrow \text{Sets}$$
$$M \longmapsto \text{collection of families } \{Q_m\}_{m \in M}$$

Pullback = base change

$\mathcal{F}$ is a presheaf on "spaces"

Moduli space $\mathcal{M}$ if

$$\mathcal{F} = \mathcal{F}_{\mathcal{M}} \qquad \mathcal{F}_{\mathcal{M}}(M) = \mathrm{Map}(M, \mathcal{M})$$

Kubota: "spaces" = Man   objects: smooth mflds
morphisms: smooth maps

Q = qtm lattice systems

<u>Sheaves on Man</u> $\mathcal{F}: \mathrm{Man}^{op} \longrightarrow \mathrm{Set}$

<u>Examples</u>: $\mathcal{F}_m(M) = C^\infty(M, m) = \mathrm{Man}(M, m)$

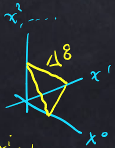
$N_1, N_2 \in \mathrm{Man}$: $\mathcal{F}(M) = C^\infty(M \times N_1, N_2)$

$q \in \mathbb{Z}$ $\mathcal{F}(M) = \Omega^q(M)$

Idea to construct $|\mathcal{F}|$ geometric realization

X top space

$S^q(X) = C^0(\Delta^q, X)$



Δ^q: $\sum x^i = 1$
$0 \le x^i \le 1$

$S^\cdot(X)$: $S^0 \rightleftarrows S^1 \rightleftarrows S^2 \; \cdots$ simplicial set

Given $S^\cdot$ simplicial set, form

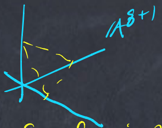
$$|S^\cdot| = \coprod_{q=0}^{\infty} S^q \times \Delta^q \Big/ \sim$$

$$\varphi_X : |S^\cdot(X)| \longrightarrow X$$

<u>Thm (Milnor)</u>: φ_X is a hty equivalence

Given $\mathcal{F} : \mathrm{Man}^{op} \longrightarrow \mathrm{Set}$

$\{ \mathcal{F}(\Delta^{\bullet}_{ext}) \}$ is a simplicial set



$\Delta^{\bullet}_{ext} = \{ \sum x^i = 1 \}$

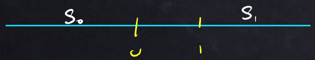
$\cong \mathbb{A}^{\bullet}$

Define $|\mathcal{F}|$ to be the geometric realization of $\{ \mathcal{F}(\Delta^{\bullet}_{ext}) \}$

Rmk: $|\mathcal{F}_M|$ has the hty type of M.

Q: What is $|\mathcal{F}|$? $[M, |\mathcal{F}|]$?

Defn: $s_0, s_1 \in \mathcal{F}(M)$ are concordant / homotopic if $\exists s \in \mathcal{F}(\mathbb{R} \times M)$



$\mathcal{F}[M]$ = homotopy classes in $\mathcal{F}(M)$

Warning: $M \longmapsto \mathcal{F}[M]$ typically not a sheaf

Thm (Madsen-Weiss): $\mathcal{F}[M] \longrightarrow [M, |\mathcal{F}|]$ iso

Construction of map:

$$\mathcal{F}(M) \overset{\text{Yoneda}}{\cong} \mathrm{Map}(\mathcal{F}_M, \mathcal{F}) \xrightarrow{|\cdot|} [|\mathcal{F}_M|, |\mathcal{F}|] \overset{\text{Milnor}}{\xleftarrow{\cong}} [M, |\mathcal{F}|]$$

Factors through $\mathcal{F}[M]$:

$$|\mathcal{F}_{\mathbb{R}\times M}| \simeq |\mathcal{F}_{\mathbb{R}}| \times |\mathcal{F}_M| \simeq \mathbb{R} \times M \longrightarrow |\mathcal{F}|$$